Preferences, Binary Relations, and Utility Functions

Including also:

The General Theory of Choice

Equivalence Relations and Partitions

Preferences, Binary Relations, and Utility Functions

Suppose we continue to assume that a particular consumer's preference is described by a utility function, for example $u(x_1, x_2) = x_1 x_2$. We define several sets associated with $u(\cdot)$, called binary relations.

Let $R_u := \{ \ (x', x) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \mid u(x') \geqq u(x) \ \}$. We generally write $x' R_u x$ or $x' \succsim_u x$ instead of $(x', x) \in R_u$ — thus, $x' \succsim_u x$ if and only if $u(x') \geqq u(x)$.

We also define the sets P_u, $\succ_u$, I_u, and $\sim_u$ as follows:

$$
\begin{aligned}
x' P_u x \quad &\Leftrightarrow \quad [x' R_u x \text{ and } \textit{not } x R_u x']; \qquad \succ_u = P_u \ ; \\
x' I_u x \quad &\Leftrightarrow \quad [x' R_u x \text{ and } x R_u x']; \qquad \sim_u = I_u \ .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_u \quad &= \quad \{(x', x) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \mid u(x') > u(x)\} \ ; \\
I_u \quad &= \quad \{(x', x) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \mid u(x') = u(x)\} \ .
\end{aligned}
$$

We say

"the decision maker (strictly) prefers x' to x if $x' P_u x$;
"the decision maker is indifferent between x' and x if $x' I_u x$;
"the decision maker (weakly) prefers x' to x if $x' R_u x$.

Note that $R_u = P_u \cup I_u$ and $P_u \cap I_u = \emptyset$.

The sets R_u, P_u, and I_u are called *binary relations* between members of $\mathbb{R}^l_+$ (or "on $\mathbb{R}^l_+$"). Binary relations are important, and they need not come from utility functions, as we've defined them here; that's just one way a binary relation can arise. More generally, a binary relation is simply a set of ordered pairs.

Definition: Let X and Y be sets. A **binary relation** between members of X and members of Y is a subset of $X \times Y$ — *i.e.,* is a set of ordered pairs $(x, y) \in X \times Y$.

Notation: For a relation $R \subseteq X \times Y$ we often write xRy instead of $(x, y) \in R$, just as we have done above for the relations R_u, P_u, and I_u.

Examples: Some examples of binary relations are provided in an appendix.

Binary relations that *do* come from utility functions have some characteristics that are important. For the following, continue to assume that $u(\cdot)$ is a utility function.

(1) R_u is **complete** — *i.e.*, for all $x, x' \in \mathbb{R}^l_+$: either $x' \succsim_u x$ or $x \succsim_u x'$.

(2) R_u, P_u, and I_u are all **transitive** — *i.e.*, for all $x, x', x'' \in \mathbb{R}^l_+ : (x'' \succsim_u x' \;\&\; x' \succsim_u x) \Rightarrow x'' \succsim_u x$, and similarly for P_u, and I_u.

(3) If $u(\cdot)$ is continuous, then R_u is **continuous** — *i.e.*, all its upper and lower contour sets are closed sets. That is, for each $x \in X$, the **upper contour set** $R_u x := \{x' \in \mathbb{R}^l_+ \mid x' R_u x\}$ and the **lower contour set** $x R_u := \{x' \in \mathbb{R}^l_+ \mid x R_u x'\}$ are both closed sets.

Remark: Clearly, an equivalent definition of continuity of a relation is that each of its *strict* upper and lower contour sets, $P_u x := \{x' \in \mathbb{R}^l_+ \mid x' P_u x\}$ and $x P_u := \{x' \in \mathbb{R}^l_+ \mid x P_u x'\}$, is an *open* set. You should see why this is so.

Exercise: Provide a proof for each of the propositions (1), (2), and (3). For (3), you might find it easier to begin by proving it for a simple specific utility function like $u(\mathbf{x}) = x_1 x_2$ or $u(\mathbf{x}) = x_1 + x_2$.

What the propositions (1) - (3) tell us is that if a consumer makes choices according to a utility function, then his preference will display some regularities — it will be "well behaved," we might say. This raises the question whether the converse is true: if a consumer's preference is well behaved, can he be represented as making choices according to a utility function? This is an important question because functions, especially continuous functions, are much more tractable analytically than binary relations. The Representation Theorem given below — a central result in the general theory of choice — ensures that the answer to the question is "yes."

Definintion: Let R be a binary relation on a set X. A real-valued function $u : X \to \mathbb{R}$ is a **utility function** for R, or a **representation** of R, if

$$\forall x, x' \in X : u(x') \geqq u(x) \Leftrightarrow x' R x. \tag{*}$$

R is said to be **representable** if there is a utility function for R.

Remark: If $R = P \cup I$ and $P \cap I = \emptyset$, then the condition (*) is equivalent to

$$\forall x, x' \in X : u(x') > u(x) \Leftrightarrow x' P x. \tag{**}$$

Representation Theorem: If a relation R on a set $X \subseteq \mathbb{R}^l_+$ is complete, transitive, and continuous, then it is representable. Moreover, it is representable by a *continuous* utility function.

Proof: Debreu, on page 56, Proposition (1), gives a proof. Jehle & Reny, on page 120, Theorem 3.1, give a proof for relations that are complete, transitive, continuous, and strictly increasing.

Do we really need all three assumptions, or "axioms," about a preference in order to know that it is representable by a utility function? For example, it seems plausible that if we don't insist that the utility function be continuous, we may be able to at least ensure that a (possibly discontinuous) representation of R exists if we at least know that R is complete and transitive. Or perhaps if R satisfies one or more additional assumptions as well, but assumptions that are not as strong as continuity, then *that* will be enough to ensure that R is representable. What we want in this kind of situation is a collection of *counterexamples*: for each assumption in our theorem, we want an example that demonstrates that if all the remaining assumptions are satisfied, but that one isn't, then R need not be representable. One such counterexample is given below: a relation R that is complete and transitive, but is not continuous, and for which no utility function exists.

Exercise: Provide counterexamples to show that neither completenes nor transitivity can be dispensed with in the theorem above.

Example (Lexicographic Preference): This is an example of a preference relation — a relation which is both complete and transitive — which is not representable. Of course, it's not a *continuous* relation; otherwise we would have a counterexample to the truth of the theorem. Let $\succsim$ on $\mathbb{R}^2_+$ be defined by

$$(x_1', x_2') \succsim (x_1, x_2) \Leftrightarrow [x_1' \geqq x_1 \text{ or } (x_1' = x_1 \text{ \& } x_2' \geqq x_2)].$$

See Figure 1. You should be able to show that this relation is not continuous. How would we show that it's not representable? We have to show that *no* utility function could represent $\succsim$ — not so easy, as it turns out. The proof relies on a fairly deep mathematical result: that the set of all real numbers (or any non-degenerate real interval) is an "uncountable" set. If we accept that mathematical fact, then the proof is not so bad: we assume that $\succsim$ has a representation $u(\cdot)$, and then we use that to establish that $\mathbb{R}_+$ is countable, which we know to be false. Therefore, our assumption that $\succsim$ has a representation $u(\cdot)$ cannot be correct. This is called an *indirect proof*, or a *proof by contradiction*.

Thus, we assume that $u(\cdot)$ is a utility function for $\succsim$. For each $x \in \mathbb{R}_+$, define the two real numbers $a(x) = u(x, 1)$ and $b(x) = u(x, 2)$ (see Figure 2). Clearly, $a(x) < b(x)$ for each x, and therefore, for each x, there is a rational number $r(x)$ that lies between $a(x)$ and $b(x)$. Moreover, if $x < \tilde{x}$, then

$r(x) < b(x)$ and $b(x) < a(\tilde{x})$ and $a(\tilde{x}) < r(\tilde{x})$; we therefore have $r(x) < r(\tilde{x})$ whenever $x < \tilde{x}$ — in particular, $x \neq \tilde{x} \Rightarrow r(x) \neq r(\tilde{x})$, so that $r(\cdot)$ is a *one-to-one* mapping from $\mathbb{R}_+$ to a subset of the set $\mathbb{Q}$ of rational numbers. Since any subset of $\mathbb{Q}$ is countable, this implies that $\mathbb{R}_+$ is countable, which we know is false. ■

Relations that are both complete and transitive clearly have a kind of "ordering" feature about them. This is important enough that this kind of relation is given a name:

Definition: A **preorder**, or **preference**, on a set X is a complete and transitive binary relation on X. We often write $\succsim$ instead of R for a preorder.

There are a number of other properties that the preference R_u described by $u(x_1, x_2) = x_1 x_2$ has in common with lots of other preferences, properties that have important implications.

(4) A preference $\succsim$ on a set $X \subseteq \mathbb{R}^l$ is **monotone** (or **increasing**) if $\mathbf{x} \geqq \mathbf{x}' \Rightarrow \mathbf{x} \succsim \mathbf{x}'$.

(5) A preference $\succsim$ on $X \subseteq \mathbb{R}^l$ is **strictly monotone** (or **strictly increasing**) if ($\mathbf{x} \geqq \mathbf{x}'$ & $\mathbf{x} \neq \mathbf{x}'$) $\Rightarrow \mathbf{x} \succ \mathbf{x}'$.

Note that if a preference on $X \subseteq \mathbb{R}^l$ is strictly increasing, then its indifference sets are surfaces in $\mathbb{R}^l$, and if $n = 2$ then they are curves, and these indifference curves slope downward. In fact, the indifference sets are surfaces (but do not necessarily slope downward) for a much broader class of preferences than just the increasing ones:

(6) A preference $\succsim$ on $X \subseteq \mathbb{R}^l$ is **locally nonsatiated (LNS)** if for every $\mathbf{x} \in X$, every neighborhood of $\mathbf{x}$ contains a bundle strictly preferred to $\mathbf{x}$ — *i.e.*, if $\forall \mathbf{x} \in \mathbf{X} : \forall \epsilon > \mathbf{0} : \exists \mathbf{x}' \in X : (|\mathbf{x}' - \mathbf{x}| < \epsilon$ & $\mathbf{x}' \succ \mathbf{x})$.

Exercise: Show that if $\alpha > 0$ then the preference described by the utility function $u(x, y) = y - \alpha(x - \beta)^2$ on $\mathbb{R}^2_+$ is not increasing but is locally nonsatiated. You should draw several of the indifference curves for this preference and try to determine what its regularities are.

(7) A preference $\succsim$ on $X \subseteq \mathbb{R}^l$ is **strictly convex** if whenever $\bar{\bar{\mathbf{x}}} \neq \bar{\mathbf{x}}$ & $\bar{\bar{\mathbf{x}}} \succsim \bar{\mathbf{x}}$ & $0 < t < 1$, then $(1 - t)\ \bar{\mathbf{x}} + \mathbf{t}\bar{\bar{\mathbf{x}}} \succ \bar{\mathbf{x}}$.

Remark: If $u(\cdot)$ is a utility function for the preference $\succsim$, then $\succsim$ is strictly convex if and only if $u(\cdot)$ is strictly quasiconcave — *i.e.,* if and only if whenever $\bar{\bar{\mathbf{x}}} \neq \bar{\mathbf{x}}$ & $\bar{\bar{\mathbf{x}}} \succsim \bar{\mathbf{x}}$ & $0 < t < 1$, then $u((1 - t)\bar{\mathbf{x}} + t\bar{\bar{\mathbf{x}}}) > u(\bar{x})$.

(8) A preference $\succsim$ on $X \subseteq \mathbb{R}^l$ is **convex** if whenever $\bar{\bar{\mathbf{x}}} \succsim \bar{\mathbf{x}}$ & $0 \leqq t \leqq 1$, then $(1-t)\bar{\mathbf{x}} + t\bar{\bar{\mathbf{x}}} \succsim \bar{\mathbf{x}}$.

Remark: If $u(\cdot)$ is a utility function for the preference $\succsim$, then $\succsim$ is convex if and only if $u(\cdot)$ is quasiconcave — *i.e.,* if and only if whenever $\bar{\bar{\mathbf{x}}} \succsim \bar{\mathbf{x}}$ & $0 \leqq t \leqq 1$, then $u((1-t)\bar{\mathbf{x}} + t\bar{\bar{\mathbf{x}}}) \geq u(\bar{x})$.
Exercise: Provide a proof.

Exercise: Show that the preference described by the utility function $u(x_1, x_2) = \alpha x_1 + \beta x_2$ on $\mathbb{R}^2_+$ is convex but not strictly convex. Do the same for the utility functions $u(x_1, x_2) = \min(2x_1, x_2)$ and $u(x_1, x_2) = \min(4x_1 + 8x_2, 10x_1 + 5x_2)$. You should draw one of the indifference curves for each of these preferences. Which of these utility functions are strictly increasing?

Remark: A preference is convex if and only if every one of its upper contour sets is a convex set.
Exercise: Provide a proof.

The General Theory of Choice

Demand theory deals with decision making in markets by price-taking individuals: the set from which the consumer must choose is his budget set, those bundles in $\mathbb{R}^l_+$ that satisfy the linear budget constraint $p \cdot \mathbf{x} \leqq w$. We would like to have a theory that includes the "linear budget constraint" case but is also general enough to include lots of other situations too. That's what the Theory of Choice does. We assume just that X is a set (of *alternatives*); B is a subset of X (the decision maker's *budget set*, or *feasible set,* or *constraint set*); and R is a preorder on X (the decision maker's *preference*). Note that X need not be a Euclidean space, or even a linear, metric, or topological space, and that B need not be defined by linear constraints.

Definition: Let R be a preorder defined on a set B. An element z of B is said to be R**-maximal**, or simply **maximal,** in B if there is no element x of B for which xPz.

The decision maker's **preference maximization problem (PMP)** is the following:

$$\textbf{(PMP)} \qquad \text{Choose an } x \in B \text{ which is } R\text{-maximal in } B.$$

Note that the (CMP) is simply the (PMP) for the special case in which $X = \mathbb{R}^l_+$ and B is defined by nonnegativity constraints and a linear budget constraint. We ask the same questions about the (PMP) and its solutions as we do about the (CMP):

(a) Under what conditions does (PMP) have a solution?

(b) Under what conditions is the solution unique?

(c) How does the solution respond to changing conditions?

We can give some fairly general answers to at least the first two questions. Note, however, that these aren't the *only* answers — they simply provide us with *some* conditions under which the (PMP) will have a solution and under which a solution will be unique. But they *are* pretty general conditions.

Existence Theorem: If R is continuous and B is compact, then (PMP) has a solution.

Proof: The proof is left as an exercise. You should apply the Bolzano-Weierstrass Theorem (see Simon & Blume).

Uniqueness Theorem: If B is convex and R is strictly convex, then (PMP) has at most one solution.

Proof: The proof is left as an exercise.

Note that the Existence Theorem assumes that continuity and compactness have a meaning — thus, that X is a metric space, or at least a topological space. And the Uniqueness Theorem assumes that convexity has a meaning — thus, that X is a linear space (*i.e.*, a vector space), although not necessarily finite-dimensional or with any topological structure. And finally, note that both theorems apply to the (CMP), in which X is a subset of a Euclidean space and B is a convex set. (**Exercise:** Show that the consumer's budget set in the (CMP) is a convex set.)

For a more general treatment, in which similar results are obtained under much weaker assumptions (*i.e.*, the preference relation is assumed to satisfy much weaker conditions than transitivity and completeness), see Chapter 7 of Border (Kim Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, 1985 paperback).

Representing a preference by a utility function is often just as useful in the general theory of choice as it is in the special case of demand theory. Note that the definition of a utility function representation that we gave in the preceding section applies without change to the general theory: the definition placed no restrictions on the set X. Although demand theory is about consumption sets X consisting of bundles in $\mathbb{R}^l_+$, we did not assume this in the definition. The Representation Theorem did assume that X is a subset of $\mathbb{R}^l_+$; however, the theorem is actually true for any set X in which open and closed sets are well defined — *i.e.*, any set X that has a topological structure (such as a metric space). Debreu's statement of the theorem, and his proof, are in this more general topological setting.

We already know, of course, that representation of a given preference is not unique. Different utility functions can represent the same preference — for example, a Cobb-Douglas utility function and its logarithmic form. The following definition allows us to characterize *all* the utility representations of a preference relation.

Definition: Let $u : X \to \mathbb{R}$ be a real-valued function defined on the set X. The function $\widetilde{u} : X \to \mathbb{R}$ is an **order-preserving transformation** of u if there is a strictly increasing real function $f : u(X) \to \mathbb{R}$ such that $\widetilde{u} = f \circ u$ — *i.e.*, if for every $x \in X$, $\widetilde{u}(x) = f(u(x))$.

Remark: Two real-valued functions u and $\tilde{u}$ on a set X are order-preserving transformations of one another if and only if they are both representations of the same relation R on X.

Equivalence Relations and Partitions

We have already remarked that if a preference relation on $\mathbb{R}^2_+$ is strictly increasing, then its indifference sets are downward sloping indifference *curves* in $\mathbb{R}^2_+$. The indifference curves form a "partition" of $\mathbb{R}^2_+$ — they divide it up into mutually disjoint sets that together cover all of $\mathbb{R}^2_+$. When we move to a different choice space than $\mathbb{R}^2_+$, the indifference sets are no longer curves, but they still partition the choice space. This is because the indifference relation is an "equivalence relation." These ideas appear again and again in economics (and in other mathematical contexts), and we formalize them here.

Definition: A relation R on a set X is

(a) **Reflexive** if $\forall x \in X : xRx$;

(b) **Symmetric** if $\forall x, x' \in X : xRx' \Rightarrow x'Rx$.

Definition: An **equivalence relation** on a set X is a relation that is reflexive, symmetric, and transitive.

We often denote an equivalence relation by the symbol $\sim$, and say that "x is equivalent to x' " if $x \sim x'$. The set $\{x \in X \mid x \sim \bar{x}\}$ of all members of X that are equivalent to a given member $\bar{x}$ is called the **equivalence class** of $\bar{x}$ and is often written $[\bar{x}]$.

Examples: If X is a set of people, the relation "is the same age as" is an equivalence relation. The relation "is a brother of" is not. The relation "lives in the same house as" is an equivalence relation. The relation "lives next door to" is not. If $f : X \to Y$ is a function mapping a set X into a set Y, then the relation $\sim$ defined by $x \sim x' \Leftrightarrow f(x) = f(x')$ is an equivalence relation; indeed, if $Y = \mathbb{R}$ then f could be interpreted as a utility function on X, and then the equivalence classes for this equivalence relation are the indifference sets for f.

Notice that each of the above examples of an equivalence relation "partitions" the set X into the relation's equivalence classes. We define a partition formally as follows:

Definition: A **partition** of a set X is a collection $\mathcal{P}$ of subsets of X that satisfies the two conditions

(1) $A, B \in \mathcal{P} \Rightarrow (A = B \text{ or } A \cap B = \emptyset)$;

(2) $\bigcup_{A \in \mathcal{P}} A = X$.

Informally, a partition of X is a collection of subsets that are *mutually exclusive* and *exhaustive*. Every member of X is in one and only one member of $\mathcal{P}$.

Remark: If $\sim$ is an equivalence relation on a set X, then the collection of its equivalence classes is a partition of X. Conversely, if $\mathcal{P}$ is a partition of X, then the relation $\sim$ defined by $x \sim x'$ $\Leftrightarrow \exists S \in \mathcal{P} : x, x' \in S$ is an equivalence relation, and its equivalence classes are the elements of the partition. **Exercise:** Provide a proof.

Remark: If R is a preference relation (a complete and transitive relation) on a set X, then its indifference relation is an equivalence relation and its indifference sets are the equivalence classes, which form a partition of X. **Exercise:** Provide a proof.

Example: Let X be a set and let U be the set of all real-valued functions on X. Define the relation $u \sim \tilde{u}$ on U as follows: $u \sim \tilde{u}$ if $\tilde{u}$ is an order-preserving transformation of u. Then $\sim$ is an equivalence relation on U. An equivalence class $[u]$ consists of all the utility functions on X that represent the same preference as u. **Exercise:** Prove that $\sim$ is an equivalence relation.

Appendix:
Some Examples of Binary Relations

Example 1:

X is a set of students, say $X = \{A, B, C, D\}$.
Y is a set of courses, say $Y = \{501A, 501B, 520\}$.

Then $X \times Y$ has 12 elements, as depicted in Figure 3.
One relation $R \subseteq X \times Y$ is the set of pairs (x, y) for which "x is enrolled in y."

Example 2:

X is a set of people.
Each of the following is a binary relation on X (*i.e.*, is a subset of $X \times X$):

(a) xNy : x lives next door to y. N would typically be symmetric, irreflexive, and not transitive.

(b) xBy : x lives on the same block as y. B would typically be reflexive, symmetric, and transitive — an equivalence relation.

(c) xSy : x is a sister of y. S would typically be irreflexive but not symmetric or transitive (unless all elements of S are female).

(d) xAy : x is an ancestor of y. A would typically be irreflexive, asymmetric, and transitive — a strict preorder.

(e) xMy : x is the mother of y. M would be irreflexive, asymmetric, and not transitive.

(f) xFy : x is in the same family as y. F would typically be reflexive, symmetric, and transitive — an equivalence relation.

Example 3:

X and Y are arbitrary sets and $f : X \to Y$ is a function from X to Y.

Define G as follows: $G := \{\ (x, y) \in X \times Y \mid y = f(x)\ \}$. Clearly, G is simply the graph of the function f. But we can just as well regard G as a relation between members of X and Y, where xGy means that $y = f(x)$.

Define $x \sim x'$ to mean $f(x) = f(x')$. Then $\sim$ is reflexive, symmetric, and transitive — an equivalence relation on X.

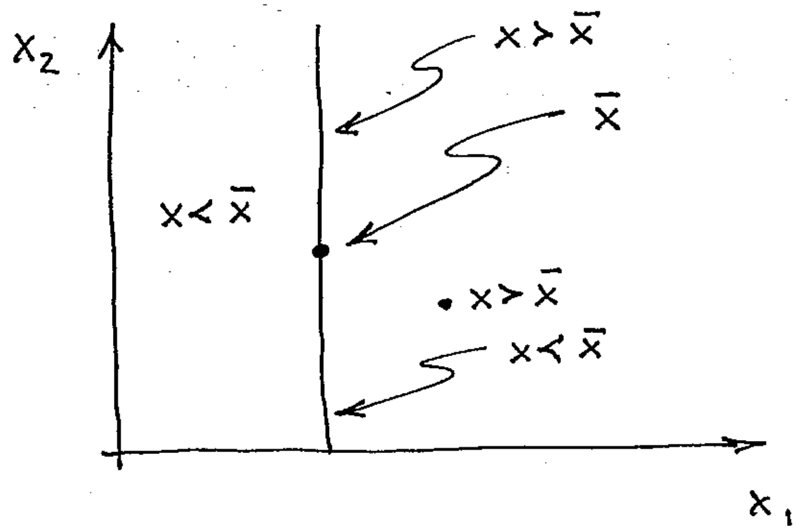

FIGURE 1: LEXICOGRAPHIC PREFERENCE

NOTE THAT ALL INDIFFERENCE SETS ARE SINGLETONS: $x \sim \bar{x} \Rightarrow x = \bar{x}$.

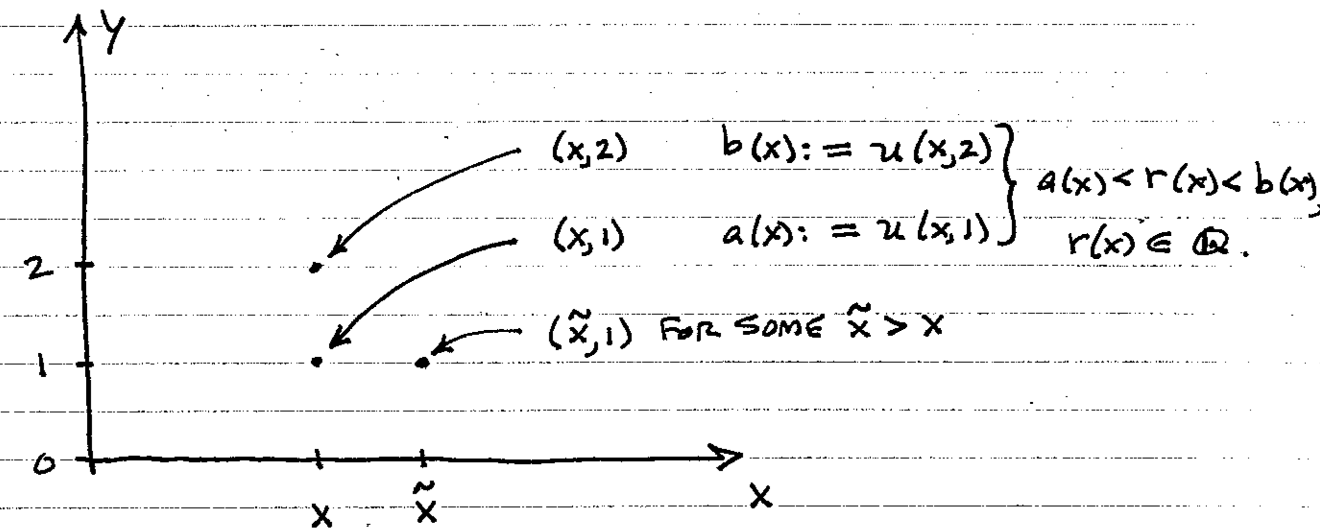

FIGURE 2:

CONSTRUCTION OF A ONE-TO-ONE FUNCTION FROM $\mathbb{R}_+$ TO $\mathbb{Q}$.

(1) $a(x) < r(x) < b(x), \quad \forall x \in \mathbb{R}_+$.

(2) $b(x) < a(\tilde{x}) \underset{\text{BY (1)}}{<} r(\tilde{x})$, IF $x < \tilde{x}$.

$\therefore \quad x < \tilde{x} \Rightarrow r(x) < r(\tilde{x})$.

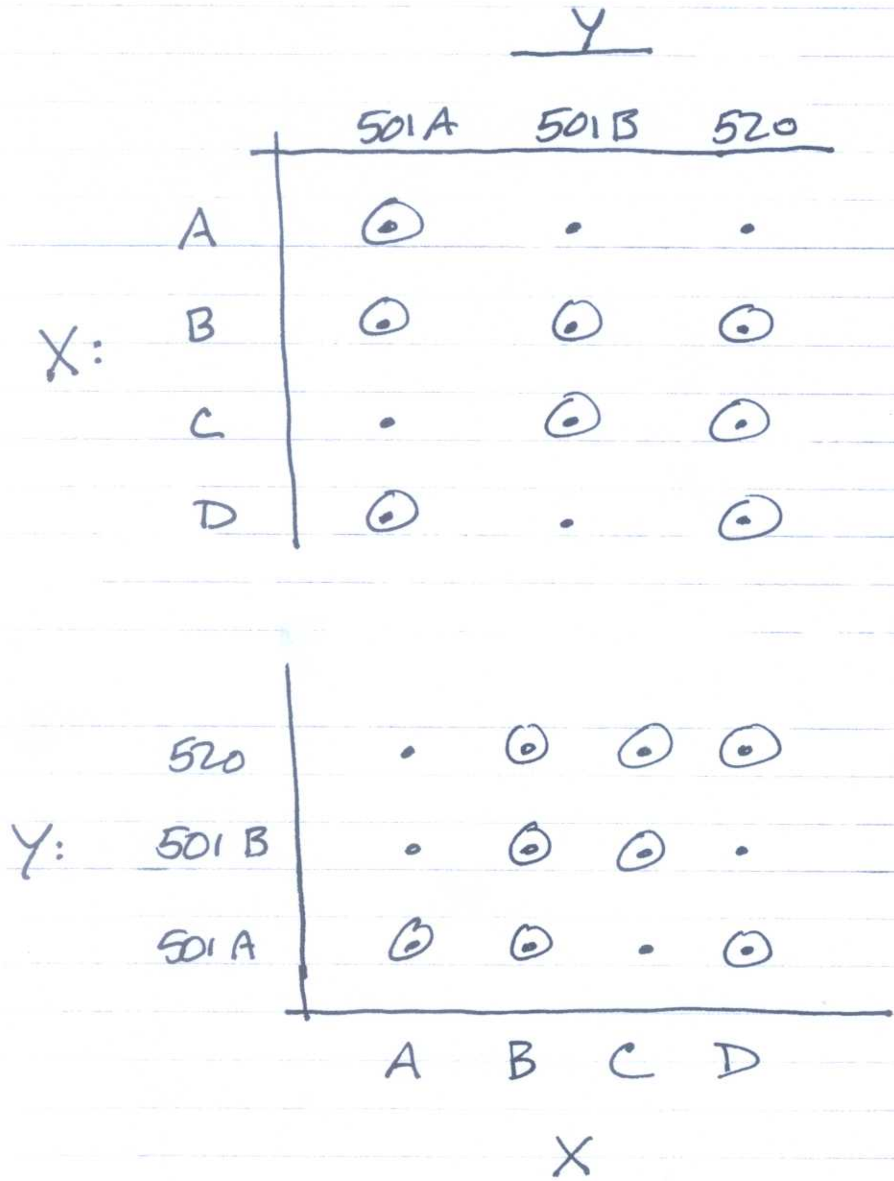

FIGURE 3

TWO WAYS OF DRAWING $X \times Y$
AND $R \subseteq X \times Y$.